27 Questions about the Cubic Surface

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December 21, 2018

What follows is a working document. It will evolve over time. In spite of two centuries of research on cubic surfaces, it appears that there are still many unresolved questions, especially when it comes to computational, tropical and applied aspects. Please feel free to circulate this text. It is aimed at stimulating further work on cubic surfaces, or equivalently, on symmetric $4 \times 4 \times 4$ tensors. Your feedback and comments will be greatly appreciated.

A cubic surface in $\mathbb{P}^3$ is the zero set of a homogeneous polynomial

$$\begin{aligned} f \;=\; & c_1x^3 + c_2y^3 + c_3z^3 + c_4w^3 + c_5x^2y + c_6x^2z + c_7x^2w + c_8xy^2 + c_9y^2z + c_{10}y^2w + c_{11}xz^2 \\ & + c_{12}yz^2 + c_{13}z^2w + c_{14}xw^2 + c_{15}yw^2 + c_{16}zw^2 + c_{17}xyz + c_{18}xyw + c_{19}xzw + c_{20}yzw. \end{aligned}$$

We work over a field K of characteristic 0, such as $\mathbb{Q}$, $\mathbb{Q}_p$, $\mathbb{Q}(t)$, $\mathbb{R}$, $\mathbb{C}$, $\overline{\mathbb{Q}(t)}$, $\mathbb{C}\{\{t\}\}$. The 15-dimensional group PGL(4) acts naturally on the projective space $\mathbb{P}^{19} = \mathbb{P}(\mathrm{Sym}_3(K^4))$ whose coordinates are $(c_1{:}c_2{:}\cdots{:}c_{20})$. Our first question concerns the orbits of that action. Here the point of departure would be Kazarnovskii's general formula for degrees of orbits.

Question 1. *Given a generic cubic f, what can we say about the orbit closure $\overline{\mathrm{PGL}(4) \cdot f}$? What is the degree of this variety? Can we determine some defining polynomial equations?*

The geometric invariant theory of cubic surfaces is well understood. Salmon (1860) found six fundamental invariants, of degrees 8, 16, 24, 32, 40 and 100. The square of the last one is a polynomial in the first five. Beklemishev (1982) proved that Salmon's list is complete.

Question 2. *How to evaluate the six invariants? Same question for their tropicalizations.*

One obvious invariant is the discriminant of f, a polynomial of degree 32 in $c_1, \ldots, c_{20}$. Edge (1980) corrects a formula written in fundamental invariants due to Salmon and Clebsch.

Question 3. *How many monomials does the discriminant have? How many vertices does its Newton polytope have, i.e. how many D-equivalence classes of regular triangulations?*

The discriminant of f equals the resultant of the four quadrics $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}, \frac{\partial f}{\partial w}$.

Question 4. *Can we write the resultant of four quaternary quadrics as the determinant of an 8×8-matrix whose entries are linear forms in the brackets? This is the Chow form of the Veronese surfaces in $\mathbb{P}^9$. Such a formula is derived from a nice Ulrich sheaf on the surface.*

Assuming the answer to Question 4 to be affirmative, we specialize to get an 8×8-matrix whose entries are quartics in $c_1, \ldots, c_{20}$ and whose determinant equals the discriminant of f.

Question 5. *Which varieties in* $\mathbb{P}^{19}$ *arise by imposing rank conditions on this* 8×8*-matrix?*

E.J. Nansen (1899) writes the above resultant as the determinant of a 20×20-matrix. We can again specialize this to a matrix whose determinant is the discriminant of f.

Question 6. *Which varieties in* $\mathbb{P}^{19}$ *arise by imposing rank conditions on this* 20×20*-matrix?*

It seems reasonable to surmise that the loci in Questions 5 and 6 are cubic surfaces with prescribed types of singularities. The simplest scenario is the occurrence of ≤ 4 simple nodes.

Question 7. *For* $k = 2, 3, 4$*, the variety of* k*-nodal cubics is irreducible of codimension* k *in* $\mathbb{P}^{19}$*. Sascha Timme computed that the degrees of these varieties are* 280*,* 800 *and* 305 *respectively. Can we find explicit low-degree polynomials that vanish on these varieties?*

Question 8. *Can we find* 17 *real points in* $\mathbb{P}^3$ *such that all* 280 *of the* 2*-nodal cubics through these points are real? Can we find* 16 *real points in* $\mathbb{P}^3$ *such that all* 800 *of the* 3*-nodal cubics through these points are real? Also, are there configurations such that no such cubic is real?*

The 4-nodal cubics are Cayley symmetroids. These arise in semidefinite programming.

Question 9. *Can we find* 15 *real points in* $\mathbb{P}^3$ *that lie on* 305 *real Cayley symmetroids?*

The following question arose from a conversation with Hannah Markwig.

Question 10. *Can the numbers* 280*,* 800 *and* 305 *be derived tropically?*

We now refer to the construction of cubic surfaces by blowing up $\mathbb{P}^2$ in six points.

Question 11. *How to construct six points with integer coordinates in* $\mathbb{P}^2$*, and a basis for the space of cubics through these points, such that the resulting cubic surface in* $\mathbb{P}^3$ *has a smooth tropical surface for its* 2*-adic tropicalization? Which unimodular triangulations arise?*

Up to symmetry, there are 14373645 unimodular triangulations of the tetrahedron $3\Delta_3$.

Question 12. *Given a cubic surface over a valued field* K*, how to decide whether its tropicalization is smooth after some linear change of coordinates? How to search* $\mathrm{PGL}(4, K)$*?*

Salmon's invariant of degree 100 vanishes precisely when the cubic surface has an Eckhard point, that is, a point common to three of its 27 lines. This invariant deserves further study.

Question 13. *What is the singular locus of the Eckhard hypersurface of degree* 100 *in* $\mathbb{P}^{19}$*?*

The cubic polynomial f can be interpreted as a symmetric tensor of format $4 \times 4 \times 4$. A typical tensor has complex rank 5, but its real rank becomes 6 as one crosses the *real rank boundary.* This is a hypersurface of degree 40 in $\mathbb{P}^{19}$ studied by Michalek and Moon (2018).

Question 14. *What can be said about the tropicalization of the Michalek-Moon hypersurface?*

Seigal (2018) identifies the Hessian discriminant as a locus where the complex rank of f jumps from 5 to 6. This is a hypersurface of degree 120 in $\mathbb{P}^{19}$, invariant under $\mathrm{PGL}(3, K)$.

Question 15. *How to write the Hessian discriminant in terms of fundamental invariants?*

The *eigenpoints* of f are the fixed points of the gradient map $\nabla f : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$. For generic cubics f, there are 15 eigenpoints. They form the *eigenconfiguration* of the $4 \times 4 \times 4$ tensor f.

Question 16. *Which configurations of* 15 *points in* $\mathbb{P}^3$ *arise as eigenpoints of a cubic surface?*

The *eigendiscriminant* is a hypersurface of degree 96 in $\mathbb{P}^{19}$. It represents cubic surfaces that possess an eigenpoint of multiplicity ≥ 2. This hypersuface deserves further study.

Question 17. *Does there exists a compact determinantal formula for the eigendiscriminant?*

We learned from Bruin and Sertöz (2018) that there are 255 Cayley symmetroids containing a general $(2,3)$-curve in $\mathbb{P}^3$, one for each 2-torsion point on the Jacobian. This is less than the number 305 of Cayley symmetroids found in a general $\mathbb{P}^4$ in $\mathbb{P}^{19}$; cf. Question 9.

Question 18. *What explains the drop from* 305 *to* 255 *when we count Cayley symmetroids that lie in the special* 4*-plane in* $\mathbb{P}^{19}$ *of all cubic surfaces containing a given space sextic?*

The following question paraphrases Problem 5.4 in (Sturmfels-Xu 2010). It was studied in (Bernal-Corey-DontonBury-Fujita-Merz 2017) but the authors left it largely unresolved.

Question 19. *What are all the toric degenerations of Cox rings of cubic surfaces?*

The following question paraphrases Conjecture 5.3 in (Ren-Shaw-Sturmfels 2016).

Question 20. *Can we identify a tropical basis for the universal Cox ideal of cubic surfaces?*

It is a big challenge to relate the intrinsic Del Pezzo geometry to the embedded geometry in $\mathbb{P}^3$, especially when it comes to tropical aspects. This is reminiscent from the curve case.

Question 21. *There are two generic types of tropical del Pezzo surfaces of degree* 3*, characterized by the tree arrangements in (Ren-Shaw-Sturmfels 2016). Can we identify cubics* f *that realize these two types by looking at the valuations of the six invariants in Question 2?*

The lines in $\mathbb{P}^3$ are points $p = (p_{12} : p_{13} : p_{14} : p_{23} : p_{24} : p_{34})$ in the Grassmannian $\mathrm{Gr}(2,4) \subset \mathbb{P}^5$. The *universal Fano variety* in $\mathbb{P}^{19} \times \mathbb{P}^5$ parametrizes lines on cubic surfaces. Its ideal is generated modulo the Plücker quadric by 20 polynomials of degree $(3,1)$ in (p,c).

Question 22. *Can we identify an explicit tropical basis for universal Fano variety?*

A real cubic surface in $\mathbb{P}^3_{\mathbb{R}}$ has either one or two connected components. In the latter case, the cubic is *hyperbolic*. It bounds a convex body that is of interest in optimization.

Question 23. *Can we find a semialgebraic description for the set of smooth hyperbolic cubics in* $\mathbb{P}^{19}_{\mathbb{R}}$*? How to express this case distinction in terms of the six fundamental invariants?*

Every cubic f whose Hessian discriminant (in Question 15) is non-zero has a unique representation as a sum of five third powers of linear forms, $f = \ell_1^3 + \ell_2^3 + \ell_3^3 + \ell_4^3 + \ell_5^3$. This is Sylvester's Pentahedral Theorem. Salmon (1860) uses this to write the invariants.

Question 24. *Can we find explicit linear forms $\ell_i \in \mathbb{Z}[x,y,z,w]$ such that the 2-adic tropicalization of $V(f)$ is tropically smooth. Which unimodular triangulations of $3\Delta_3$ arise?*

If we project a cubic surface $V(f)$ from a general point p on that surface then we get a double-covering of $\mathbb{P}^2$ branched along a quartic curve. The 28 bitangents of that curve are the images of the 27 lines on $V(f)$ plus one more line which is the exceptional divisor over p.

Question 25. *Can this correspondence from 27 to 28 be understand in tropical geometry? In particular, can we see the seven 4-tuples of tropical bitangents already in* $\mathrm{Trop}(V(f))$*?*

The seven 4-tuples of bitangents are explained in (Lee-Len 2017), (Chan-Jiradilik 2017). Brundu and Logar (1998) offer a computational study of cubic surfaces via the normal form

$$\begin{aligned} f \;=\; & a_1(2x^2y - 2xy^2 + xz^2 - xzw - yw^2 + yzw) \,+\, a_2(x-w)(xz+yw) \,+ \\ & a_3(z+w)(yw-xz) \,+\, a_4(y-z)(xz+yw) \,+\, a_5(x-y)(yw-xz). \end{aligned}$$

This amounts to fixing an *L-set*, which is a special configuration of five lines on $V(f)$.

Question 26. *How to compute the Brundu-Logar normal form in practise? Can we write a_1, a_2, a_3, a_4, a_5 as rational functions in $c_1, c_2, \ldots, c_{20}$? What does this tell us tropically?*

Here is another normal form, found in (Buckley-Kośir 2007). From the classical construction of *Steiner sets*, one shows that a general cubic surface has 120 distinct representations

$$f \;=\; \ell_1\ell_2\ell_3 \,+\, m_1m_2m_3,$$

where the ℓ_i and m_j are linear forms. We call this a *Steiner representation* of the cubic f.

Question 27. *How to compute Steiner representations in practise? Can it be tropicalized?*